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Weighted Markov-Type Estimates for the Derivatives of Constrained Polynomials on $[0, \infty)$

T. ERDÉLYI*

Mathematical Institute of the Hungarian Academy of Sciences, 13-15 Reáltanoda utca, Budapest H-1053, Hungary

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Throughout this paper $c_1(\cdot)$, $c_2(\cdot)$, ... will denote positive constants depending only on the values given in the parenthesis. Let Π_n be the set of all real algebraic polynomials of degree at most n. A weaker version of an inequality of the brothers Markov (see [8, 9]) asserts that

$$\max_{A \leq x \leq B} |p^{(m)}(x)| \leq \left(\frac{2n^2}{B-A}\right)^m \max_{A \leq x \leq B} |p(x)| \qquad (p \in \Pi_n;\, n, m \geq 1). \tag{1}$$

For $0 < r \leq (B-A)/2$ ($A, B \in \mathbb{R}$) let

$$D_1(A, B, r)^+ := \{z \in \mathbb{C} \mid |z-(A+r)| < r\}$$

and denote by $S_n^k(A, B, r)^+$ $(0 \leq k \leq n)$ the set of those polynomials from Π_n which have at most k roots in $D_1(A, B, r)^+$. From (40) of [2], by a simple linear transformation we obtain

THEOREM A. *Let $0 < r \leq (B-A)/2$, $A, B \in \mathbb{R}$, $0 \leq k \leq n$, $n, m \geq 1$, and $s \in S_n^k(A, B, r)^+$. Then*

$$|s^{(m)}(A)| \leq c_1(m) \left(\frac{n(k+1)^2}{\sqrt{r(B-A)}}\right)^m \max_{A \leq x \leq B} |s(x)|.$$

Let

$$\|p\|_a := \sup_{0 \leq x < \infty} |p(x) \exp(-x^a)| \qquad (p \in \Pi_n,\, a > 0), \tag{2}$$

$$D_2(r) := \{z \in \mathbb{C} \mid |z-r| < r\} \qquad (r > 0), \tag{3}$$

$$D_3(r) := \{z \in \mathbb{C} \mid \operatorname{Re} z > r\} \tag{4}$$

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and denote by $W_n^k(r)$ and $V_n^k(r)$ $(0 \leqq k \leqq n,\ r \geqq 0)$ the set of those polynomials from Π_n which have at most k roots in $D_2(r)$ and $D_3(r)$, respectively. The main purpose of this paper is to give Markov-type estimates for the derivatives of polynomials from Π_n, $W_n^k(r)$, and $V_n^0(r)$ $(0 \leqq k \leqq n,\ r > 0)$ on $[0, \infty)$ with respect to the norm $\|\cdot\|_a$. We shall prove the following theorems.

THEOREM 1. *Let $n \geqq 2$, $m \geqq 1$, and $a > 0$. Then we have*

$$\|p^{(m)}\|_a \leqq c_2(a, m)(K_n(a))^m \|p\|_a \qquad (p \in \Pi_n),$$

where

$$K_n(a) = \begin{cases} n^{2-1/a} & \text{if } \tfrac{1}{2} < a < \infty \\ \log^2 n & \text{if } a = \tfrac{1}{2} \\ 1 & \text{if } 0 < a < \tfrac{1}{2}. \end{cases}$$

THEOREM 2. *Let $n \geqq 2$, $0 \leqq k \leqq n$, $m \geqq 1$, $r \geqq 0$, and $a > 0$. Then we have*

$$\|p^{(m)}\|_a \leqq c_3(a, m)((k+1)^2 L_n(a, r))^m \|p\|_a \quad (p \in W_n^k(r)),$$

where

$$L_n(a, r) = \begin{cases} n^{2-1/a} & (0 \leqq r \leqq n^{1/a-2}) \\ \dfrac{n^{1-1/(2a)}}{\sqrt{r}} & (n^{1/a-2} \leqq r \leqq n^{1/a}) \\ n^{1-1/a} & (n^{1/a} \leqq r < \infty) \end{cases}$$

if $1 \leqq a < \infty$,

$$L_n(a, r) = \begin{cases} n^{2-1/a} & (0 \leqq r \leqq n^{1/a-2}) \\ \dfrac{n^{1-1/(2a)}}{\sqrt{r}} & (n^{1/a-2} \leqq r \leqq n^{2-1/a}) \\ 1 & (n^{2-1/a} \leqq r < \infty) \end{cases}$$

if $\tfrac{1}{2} < a \leqq 1$,

$$L_n(a, r) = \begin{cases} \log^2 n & (0 \leqq r \leqq \log^{-2} n) \\ \dfrac{\log n}{\sqrt{r}} & (\log^{-2} n \leqq r \leqq \log^2 n) \\ 1 & (\log^2 n \leqq r < \infty) \end{cases}$$

if $a = \frac{1}{2}$, *and*

$$L_n(a, r) = 1 \qquad \text{if} \quad 0 < a < \tfrac{1}{2}.$$

THEOREM 3. *If* $k = 0$, $1 \leqq m \leqq n$, *and* $0 < a \neq \frac{1}{2}$, *then up to the constant depending only on a and m Theorems* 1 *and* 2 *are sharp.*

Conjecture 1. Up to the constant $c_3(a, m)$ Theorem 2 is sharp even in the case when $k = 0$, $1 \leqq m \leqq n$, and $a = \frac{1}{2}$.

THEOREM 4. *Let* $n, m \geqq 1$, $r \geqq 0$, *and* $a > 0$. *Then we have*

$$\|p^{(m)}\|_a \leqq c_4(a, m)(G_n(a, r))^m \, \|p\|_a \qquad (p \in V_n^0(r)),$$

where

$$G_n(a, r) = \begin{cases} r^{2a-1} + n^{1-1/a} + 1 & (0 \leqq r \leqq n^{1/a}) \\ n^{2-1/a} & (n^{1/a} < r < \infty) \end{cases}$$

when $\frac{1}{2} < a < \infty$,

$$G_n(\tfrac{1}{2}, r) = \begin{cases} \log^2(r+2) & (0 \leqq r \leqq n^2) \\ \log^2(n+1) & (n^2 < r < \infty), \end{cases}$$

and

$$G_n(a, r) = 1 \qquad \text{when} \quad 0 < a < \tfrac{1}{2}.$$

THEOREM 5. *For all* $0 < a \neq \frac{1}{2}$ *and* $1 \leqq m \leqq n$, *up to the constant* $c_2(a, m)$ *Theorem* 4 *is sharp.*

Conjecture 2. Up to the constant $c_4(a, m)$ Theorem 4 is sharp even for $a = \frac{1}{2}$ and $1 \leqq m \leqq n$.

(To see this it would be sufficient to prove that Theorem 1 is sharp when $a = \frac{1}{2}$.)

Proof of Theorem 1. It is sufficient to prove the theorem when $m = 1$, from this the general case follows by induction on m. We distinguish two cases.

Case 1. $1 \leqq a < \infty$. Denote the integer part of a by $[a]$. A close inspection of its derivative shows that

$$F(x) := \left(1 - \frac{x^{[a]}}{n^{[a]/a}}\right)^{2n} \exp(x^a)$$

is monotonically decreasing in $[0, n^{1/a}]$; therefore

$$\exp(-x^a) \geqq q_{n,y}(x) := \frac{\exp(-y^a)}{(1 - y^{[a]}/n^{[a]/a})^{2n}} \times \left(1 - \frac{x^{[a]}}{n^{[a]/a}}\right)^{2n} \geqq 0 \qquad (0 \leqq y \leqq x \leqq n^{1/a}). \tag{5}$$

Now let $p \in \Pi_n$ be arbitrary. Then $s := pq_{n,y} \in \Pi_{(2[a]+1)n}$ $(0 \leqq y \leqq n^{1/a})$, so by (1) and (5) we obtain

$$\begin{aligned} |s'(y)| &\leqq \frac{2(2[a]+1)^2 n^2}{(1/2)n^{1/a}} \max_{y \leqq x \leqq y + (1/2)n^{1/a}} |p(x)\, q_{n,y}(x)| \\ &\leqq c_5(a) n^{2-1/a} \max_{y \leqq x \leqq y + (1/2)n^{1/a}} |p(x) \exp(-x^a)| \\ &\leqq c_5(a) n^{2-1/a} \|p\|_a \qquad (0 \leqq y \leqq \tfrac{1}{2} n^{1/a}). \end{aligned} \tag{6}$$

Further a simple calculation shows that

$$|q'_{n,y}(y)| \leqq c_6(a) n^{1-1/a} \exp(-y^a) \qquad (0 \leqq y \leqq \tfrac{1}{2} n^{1/a}). \tag{7}$$

Hence and from (6)

$$\begin{aligned} |p'(y) \exp(-y^a)| &= |p'(y)\, q_{n,y}(y)| \\ &\leqq |s'(y)| + |p(y)\, q'_{n,y}(y)| \\ &\leqq c_5(a) n^{2-1/a} \|p\|_a \\ &\quad + c_6(a) n^{1-1/a} |p(y) \exp(-y^a)| \\ &\leqq c_7(a) n^{2-1/a} \|p\|_a \qquad (p \in \Pi_n,\ 0 \leqq y \leqq \tfrac{1}{2} n^{1/a}). \end{aligned} \tag{8}$$

Finally by (1) we get

$$\begin{aligned} |p'(y) \exp(-y^a)| &\leqq \exp(-y^a) \frac{2n^2}{y} \max_{0 \leqq x \leqq y} |p(x)| \\ &\leqq 4n^{2-1/a} \max_{0 \leqq x \leqq y} |p(x) \exp(-x^a)| \\ &\leqq 4n^{2-1/a} \|p\|_a \qquad (p \in \Pi_n,\ \tfrac{1}{2} n^{1/a} \leqq y < \infty). \end{aligned} \tag{9}$$

Now (8) and (9) give Theorem 1 in this case.

Case 2. $0 < a \leqq 1$. We need the following Markov-type inequality,

$$\sup_{|x|<\infty} |f'(x)\exp(-|x|^b)| \leqq c_8(b)\, H_n(b) \sup_{|x|<\infty} |f(x)\exp(-|x|^b)|$$
$$(f \in \Pi_{2n},\ n \geqq 2,\ b > 0), \qquad (10)$$

where

$$H_n(b) = \begin{cases} n^{1-1/b} & \text{if} \quad 1 \leqq b < \infty \\ \log n & \text{if} \quad b = 1 \\ 1 & \text{if} \quad 0 < b < 1. \end{cases} \qquad (11)$$

(See G. Freud [4] $(2 \leqq b < \infty)$, A. L. Levin and D. S. Lubinsky [5] $(1 < b < 2)$, and P. Nevai and V. Totik [11] $(0 < b \leqq 1)$.) Now let $g \in \Pi_n$ be arbitrary and $f(x) = g(x^2) \in \Pi_{2n}$. Using (10) and the substitutions $z = x^2$ and $a = b/2$, we get

$$\begin{aligned} |g'(0)| &= \tfrac{1}{2}\, |f''(0)| \\ &\leqq c_9(b)(H_n(b))^2 \sup_{|x|<\infty} |f(x)\exp(-|x|^b)| \\ &\leqq c_{10}(a)\, K_n(a) \sup_{0 \leqq z < \infty} |g(z)\exp(-z^{a/2})| \\ &\leqq c_{10}(a)\, K_n(a)\, \|g\|_a \qquad (0 < a < \infty). \end{aligned} \qquad (12)$$

Let $p \in \Pi_n$ and $y \in [0, \infty)$ be arbitrary. Consider the polynomial $g(x) := p(x+y) \in \Pi_n$. Applying (12) to g and using that $x^a + y^a \geqq (x+y)^a$ $(x, y \geqq 0,\ 0 \leqq a \leqq 1)$, we obtain

$$\begin{aligned} |p'(y)| &= |g'(0)| \\ &\leqq c_{10}(a)\, K_n(a)\, \|g\|_a \\ &\leqq c_{10}(a)\, K_n(a) \exp(y^a) \sup_{x \geqq 0} |p(x+y)\exp(-(x+y)^a)| \\ &\leqq c_{10}(a)\, K_n(a) \exp(y^a)\, \|p\|_a, \end{aligned} \qquad (13)$$

which yields Theorem 1 in this case as well. ∎

Note 1. In case $a = 1$ Theorem 2 was proved by G. Szegö [12], but his method does not work in the general case.

Before proving Theorem 2 we establish a Bernstein-type estimate on $[0, \infty)$ with respect to the norm $\|p\|_a$.

LEMMA 1. *Let* $m \geqq 1$, $a>0$, $y>0$. *Then*

$$|p^{(m)}(y)\exp(-y^a)| \leqq c_{11}(a,m)(H_n(2a))^m y^{-m/2} \|p\|_a \qquad (p \in \Pi_n),$$

where $H_n(b)$ *is defined by* (11) *for* $b>0$.

Proof. From (10), by induction on m it is straightforward that

$$\sup_{|x|<\infty} |f^{(m)}(x)\exp(-|x|^b)| \leqq c_{12}(b,m)(H_n(b))^m \sup_{|x|<\infty} |f(x)\exp(-|x|^b)| \qquad (f \in \Pi_{2n},\ 0<b<\infty). \tag{14}$$

We prove the lemma by induction on m. The statement holds for $m=0$. Now suppose that it holds for all $0 \leqq \mu \leqq m-1$. Let $p \in \Pi_n$ be arbitrary and let $f(x) := p(x^2) \in \Pi_{2n}$. It is easy to check that with suitable constants $c_{\mu,\nu,m}$ depending only on μ, ν, and m we have

$$f^{(m)}(x) = 2^m x^m p^{(m)}(x^2) + \sum_{\substack{0 \leqq \nu \leqq \mu \leqq m-1 \\ 2\mu-\nu \leqq m}} c_{\mu,\nu,m} x^\nu p^{(\mu)}(x^2); \tag{15}$$

thus with the substitution $y=x^2$ and $b=2a$ we have

$$f^{(m)}(x)\exp(-|x|^b) = 2^m y^{m/2} p^{(m)}(y)\exp(-y^a) + \sum_{\substack{0 \leqq \nu \leqq \mu \leqq m-1 \\ 2\mu-\nu \leqq m}} c_{\mu,\nu,m} y^{\nu/2} p^{(\mu)}(y)\exp(-y^a). \tag{16}$$

Here by the induction assumption

$$\begin{aligned}
&|y^{\nu/2} p^{(\mu)}(y)\exp(-y^a)| \\
&\quad = |y^{\mu/2} p^{(\mu)} \exp(-y^b)|\, y^{(\nu-\mu)/2} \\
&\quad \leqq c_{11}(a,\mu)(H_n(2a))^\mu \|p\|_a (H_n(2a))^{\mu-\nu} \\
&\quad \leqq c_{11}(a,\mu)(H_n(2a))^m \|p\|_a \\
&\qquad (0 \leqq \nu \leqq \mu \leqq m-1,\ 2\mu-\nu \leqq m,\ y \geqq (H_n(2a))^{-2}).
\end{aligned} \tag{17}$$

Using the substitutions $y=x^2$, $b=2a$, and recalling that $f(x)=p(x^2) \in \Pi_{2n}$, from (14) we get

$$|f^{(m)}(x)\exp(-|x|^b)| \leqq c_{12}(b,m)(H_n(2a))^m \|p\|_a \qquad (y \geqq 0). \tag{18}$$

Now (16), (17), and (18) give the desired result when $y \geqq (H_n(2a))^{-2}$. If $0 < y < (H_n(2a))^{-2}$, then by Theorem 1

$$|p^{(m)}(y)\exp(-y^a)| \leqq c_2(a, m)(K_n(a))^m \|p\|_a \leqq c_2(a, m)(H_n(2a))^m y^{-m/2} \|p\|_a. \tag{19}$$

Thus the proof of the lemma is complete. ∎

Proof of Theorem 2. We distinguish three cases.

Case 1. $a \geqq 1$. We shall use the notations introduced in the proof of Theorem 1. Observe that $q_{n,y}$ $(0 \leqq y \leqq \frac{1}{2} n^{1/a})$ has all its zeros outside the circle $\{z \in \mathbb{C} \mid |z| < n^{1/a}\}$. Hence by an observation of G. G. Lorentz $q_{n,y}$ is of the form

$$q_{n,y}(x) = \sum_{j=0}^{n} a_j (x - n^{1/a})^j (n^{1/a} - x)^{n-j} \qquad \text{with all} \quad a_j \geqq 0,$$

so from Theorem B of [6], by a linear transformation we get

$$\begin{aligned} |q_{n,y}^{(j)}(y)| &\leqq c_{13}(a, j)(n^{1-1/a})^j \times \max_{y \leqq x \leqq y + (1/2)n^{1/a}} |q_{n,y}(x)| \\ &= c_{13}(a, j)(n^{1-1/a})^j \times \exp(-y^a) \qquad (0 \leqq y \leqq \tfrac{1}{2} n^{1/a},\ j \geqq 0). \end{aligned} \tag{20}$$

To prove Theorem 2 we proceed by induction on m. In case of $m = 0$ the statement is obvious. Suppose that the theorem holds for $0 \leqq j \leqq m-1$. Let $0 \leqq y \leqq r$, $n^{1/a-2} \leqq r \leqq \frac{1}{4} n^{1/a}$, and $p \in W_n^k(r)$. Then $s := pq_{n,y} \in S^k_{(2[a]+1)n}(y, y + \frac{1}{2} n^{1/a}, r/2)$, so using Theorem A and (5) we have

$$\begin{aligned} |s^{(m)}(y)| &\leqq c_{14}(a, m) \left(\frac{n^{1-1/(2a)}(k+1)^2}{\sqrt{r}} \right)^m \times \max_{y \leqq x \leqq y + (1/2)n^{1/a}} |p(x)\, q_{n,y}(x)| \\ &\leqq c_{14}(a, m)((k+1)^2 L_n(a, r))^m \times \max_{y \leqq x \leqq y + (1/2)n^{1/a}} |p(x)\exp(-x^a)| \\ &\leqq c_{14}(a, m)((k+1)^2 L_n(a, r))^m \|p\|_a \qquad (0 \leqq y \leqq r,\ n^{1/a-2} \leqq r \leqq \tfrac{1}{4} n^{1/a}). \end{aligned} \tag{21}$$

Now by (5), (20), (21), and the induction assumption we deduce

$$
\begin{aligned}
|p^{(m)}(y)\exp(-y^a)| &= |p^{(m)}(y)\, q_{n,y}(y)| \\
&\leqq |(pq_{n,y})^{(m)}(y)| \\
&\quad + \sum_{j=1}^{m} \binom{m}{j} |p^{(m-j)}(y)\, q_{n,y}^{(j)}(y)| \\
&\leqq c_{14}(a,m)((k+1)^2 L_n(a,r))^m \|p\|_a \\
&\quad + \sum_{j=1}^{m} \binom{m}{j} \exp(y^a)\, c_3(a,m-j) \\
&\quad \times ((k+1)^2 L_n(a,r))^{m-j} \|p\|_a \\
&\quad \times c_{13}(a,j)(n^{1-1/a})^j \exp(-y^a) \\
&\leqq c_{15}(a,m)((k+1)^2 L_n(a,r))^m \|p\|_a \\
&\qquad (p \in W_n^k(r),\, 0 \leqq y \leqq r,\, n^{1/a-2} \leqq r \leqq \tfrac{1}{4} n^{1/a}). \quad (22)
\end{aligned}
$$

Further by Lemma 1

$$
\begin{aligned}
&|p^{(m)}(y)\exp(-y^a)| \\
&\quad \leqq c_{16}(a,m)(H_n(2a))^m r^{-m/2} \|p\|_a \\
&\quad = c_{16}(a,m)(L_n(a,m))^m \|p\|_a \qquad (p \in \Pi_n,\, r \leqq y < \infty). \qquad (23)
\end{aligned}
$$

Now (22) and (23) give the theorem when $n^{1/a-2} \leqq r \leqq \frac{1}{4} n^{1/a}$. If $0 \leqq r \leqq n^{1/a-2}$, then Theorem 1 gives the desired result. If $\frac{1}{4} n^{1/a} \leqq r < \infty$, then using the relation $W_n^k(r) \subset W_n^k(\frac{1}{4} n^{1/a})$ and the just proved part of the theorem, we get the statement for all $r \geqq \frac{1}{4} n^{1/a}$.

Case 2. $\frac{1}{2} \leqq a \leqq 1$. We need a number of lemmas.

LEMMA 2. *For all* $n \geqq 2$ *and* $\frac{1}{2} \leqq a < \infty$ *there exist polynomials* $Q_{n,a} \in \Pi_N$ *such that*

$$
c_{17}(a) \leqq Q_{n,a}(y)\exp(y^a) \leqq c_{18}(a) \qquad (0 \leqq y \leqq n^{1/a}) \qquad (24)
$$

and

$$
1 \leqq N = N(n) := \begin{cases} [c_{19}(a)n] & \text{if } \frac{1}{2} < a < \infty \\ [(c_{19}(a)n \log n] & \text{if } a = \frac{1}{2} \end{cases} \qquad (25)
$$

hold with suitable $c_{17}(a)$, $c_{18}(a)$, *and* $c_{19}(a)$.

By using the substitutions $y=x^2$ and $b=2a$, this is a trivial consequence of the corresponding result for the interval $(-\infty, \infty)$ and weight function $\exp(-|x|^b)$ $(1 \leqq b < \infty)$; see Theorem 1.1 of [5] when $1<b<\infty$, and the proof of Theorem 3 of [11] when $b=1$.

LEMMA 3. *If* $\frac{1}{2} \leqq a < \infty$, $r>0$, $0 \not\equiv v \in \Pi_l$ *and*

$$|v(0)| \geqq c_{20}(a) \max_{0 \leqq x \leqq n^{1/a}} |v(x)| \tag{26}$$

then v *has at most* $c_{21}(a)\, ln^{-1/(2a)} \sqrt{r}$ *roots (counting multiplicities) in* $[0, r]$.

Using Lemma 1 of [2] and the substitution $x=\frac{1}{2}n^{1/a}(1+\cos t)$, we obtain Lemma 3 at once.

LEMMA 4. *If* $\frac{1}{2} \leqq a < \infty$, $n, j \geqq 0$, $r>0$, $p \in \Pi_n$ *has all its zeros in* $[2r, \infty)$ *and* $|p(0)| = \|p\|_a$, *then*

$$|p^{(j)}(0)| \leqq c_{22}(a, j) \left(\frac{M}{\sqrt{r}}\right)^j \|p\|_a,$$

where $M = Nn^{-1/(2a)}$ *and* N *is defined by* (25).

Proof. Let $\deg p = l \leqq n$ and denote the roots of p by $(2r \leqq)\, x_1 \leqq x_2 \leqq \cdots \leqq x_l\, (<\infty)$. Observe that $v := pQ_{n,a} \in \Pi_{n+N}$ satisfies (26) where $Q_{n,a}$ and N are defined by Lemma 2. With the notation

$$I_v = [2rv^4, 2r(v+1)^4] \qquad (v = 1, 2, \ldots)$$

from Lemma 3 we deduce that v and hence p as well have at most $c_{21}(a)(n+N)n^{-1/(2a)}\sqrt{2r}\,(v+1)^2$ roots (counting multiplicities) in I_v. Hence and from (25)

$$\begin{aligned}
\frac{|p^{(j)}(0)|}{\|p\|_a} &= \frac{|p^{(j)}(0)|}{|p(0)|} \leqq \left(\sum_{\mu=1}^{l} \frac{1}{x_\mu}\right)^j \leqq \left(\sum_{v=1}^{\infty} \sum_{x_\mu \in I_v} \frac{1}{x_\mu}\right)^j \\
&\leqq \left(\sum_{v=1}^{\infty} c_{21}(a)(n+N)n^{-1/(2a)} \sqrt{r}\,(v+1)^2 \frac{1}{2rv^4}\right)^j \\
&\leqq \left(\left(2\sqrt{2}\, c_{21}(a) \sum_{v=1}^{\infty} \frac{1}{v^2}\right) \frac{(n+N)\, n^{-1/(2a)}}{\sqrt{r}}\right)^j \\
&\leqq c_{22}(a, j) \left(\frac{M}{\sqrt{r}}\right)^j. \quad \blacksquare
\end{aligned}$$

LEMMA 5. *If* $\frac{1}{2} \leqq a < \infty$, $n \geqq 1$, $M^{-2} \leqq r \leqq M^2$ (*M is defined in Lemma* 4), $p \in \Pi_n$ *has all its zeros in* $[2r, \infty)$, *and* $|p(0)| = \|p\|_a$, *then*

$$|p(0)| \leqq 2\,|p(x)| \qquad \left(x \in \left[0, \frac{\sqrt{r}}{c_{23}(a)M}\right] \subset [0, 1]\right)$$

with a suitable $c_{23}(a)$.

Proof. Let $c_{23}(a) := \max[2c_{22}(a, 1), 1\}$ and

$$y := \frac{\sqrt{r}}{c_{23}(a)M}. \tag{27}$$

Since $M^{-2} \leqq r \leqq M^2$, we have

$$0 < y \leqq \min\{r, 1\}. \tag{28}$$

As $|p'(x)|$ is monotonically decreasing in $(-\infty, 2r]$, Lemma 4 implies

$$|p'(\xi)| \leqq |p'(0)| \leqq \frac{c_{22}(a, 1)M}{\sqrt{r}} \|p\|_a \qquad (0 \leqq \xi \leqq 2r). \tag{29}$$

From the mean value theorem, (27), and (28) we deduce that there exists a $\xi_1 \in (0, y)$ such that

$$\begin{aligned} |p(0) - p(y)| &= y\,|p'(\xi_1)| \\ &\leqq \frac{\sqrt{r}}{c_{23}(a)M} \frac{c_{22}(a, 1)M}{\sqrt{r}} \|p\|_a \\ &\leqq \tfrac{1}{2}\,\|p\|_a = \tfrac{1}{2}\,|p(0)|; \end{aligned} \tag{30}$$

hence

$$2\,|p(y)| \geqq |p(0)|. \tag{31}$$

As $|p(x)|$ is monotonically decreasing in $(-\infty, 2r]$, (27) and (31) give the desired result. ∎

LEMMA 6. *Let* $\frac{1}{2} \leqq a < \infty$, n, $m \geqq 1$, $M^{-2} \leqq n \leqq M^2$ (*M is defined in Lemma* 4), $s = pq$ *where* $p \in \Pi_n$ *has all its zeros in* $[2r, \infty)$, $|p(0)| = \|p\|_a$, *and* $q \in \Pi_l$. *Then*

$$|s^{(m)}(0)| \leqq c_{24}(a, m) \left(\frac{M(l+1)^2}{\sqrt{r}}\right)^m \|s\|_a. \tag{32}$$

Proof. For the sake of brevity let

$$I = I(n, a, r) := \left[0, \frac{\sqrt{r}}{c_{23}(a) M}\right] \subset [0, 1]. \tag{33}$$

Applying Markov's inequality to $q \in \Pi_l$ on I, we get

$$|q^{(m-j)}(0)| \leqq \left(\frac{2c_{23}(a) M}{\sqrt{r}} l^2\right)^{m-j} |q(x_1)| \qquad (0 \leqq j \leqq m), \tag{34}$$

where

$$x_1 \in I \text{ is such that } |q(x_1)| = \max_{x \in I} |q(x)|. \tag{35}$$

Therefore by Lemmas 4, 5, (34), and (35) we easily obtain

$$\begin{aligned} |s^{(m)}(0)| &\leqq \sum_{j=0}^{m} \binom{m}{j} |p^{(j)}(0)\, q^{(m-j)}(0)| \\ &\leqq \sum_{j=0}^{m} \binom{m}{j} c_{22}(a, j) \left(\frac{M}{\sqrt{r}}\right)^{j} \|p\|_a \left(\frac{2c_{23}(a) M l^2}{\sqrt{r}}\right)^{m-j} |q(x_1)| \\ &\leqq c_{25}(a, m) \left(\frac{M(l+1)^2}{\sqrt{r}}\right)^{m} |p(x_1)\, q(x_1)| \\ &\leqq e c_{25}(a, m) \left(\frac{M(l+1)^2}{\sqrt{r}}\right)^{m} \|s\|_a. \quad \blacksquare \end{aligned}$$

LEMMA 7. *Let* $\frac{1}{2} \leqq a < \infty$, n, $m \geqq 1$, $M^{-2} \leqq r \leqq M^2$ *(M is defined in Lemma 4), $s = pq$ where $p \in \Pi_n$ has all its zeros in $[2r, \infty)$, and $q \in \Pi_l$ has all its zeros in $\{z \in \mathbb{C} \mid 0 \leqq \operatorname{Re} z \leqq 2r\}$. Then inequality* (32) *holds.*

Proof. Because of the conditions prescribed for the roots of p and q,

$$|s(x)| \text{ is monotonically decreasing in } (-\infty, 0]. \tag{36}$$

Thus there exists exactly one $y \in (-\infty, 0]$ such that

$$|s(y)| = \|s\|_a. \tag{37}$$

Now let

$$\tilde{s}(x) := s(x + y). \tag{38}$$

Then

$$\tilde{s} = \tilde{p}\tilde{q}, \tag{39}$$

where $\tilde{p}(x)=p(x+y)\in\Pi_n$ and $\tilde{q}(x)=q(x+y)\in\Pi_l$ have all their zeros in $[2r-y, \infty)$ and $\{z\in\mathbb{C} \mid -y\leqq \operatorname{Re} z\leqq 2r-y\}$, respectively. From (36), (37), and (38) we easily deduce

$$|\tilde{s}(0)|=|s(y)|=\|s\|_a=\|\tilde{s}\|_a. \tag{40}$$

From (39) it is clear that

$$|\tilde{p}(0)|\geqq|\tilde{p}(x)|\geqq|\tilde{p}(x)\exp(-x^a)| \qquad (0\leqq x\leqq 4r-2y) \tag{41}$$

and

$$|\tilde{q}(0)|\leqq|\tilde{q}(x)| \qquad (4r-2y\leqq x<\infty). \tag{42}$$

By (39), (40), and (42) it is obvious that

$$\begin{aligned}|\tilde{p}(0)|=\frac{|\tilde{s}(0)|}{|\tilde{q}(0)|}&\geqq\frac{|\tilde{s}(x)\exp(-x^a)}{|\tilde{q}(x)|}\\ &=|\tilde{p}(x)\exp(-x^a)| \qquad (4r-2y\leqq x<\infty).\end{aligned} \tag{43}$$

Now (41) and (43) yield

$$|\tilde{p}(0)|=\|\tilde{p}\|_a. \tag{44}$$

Because of (39), $y\leqq 0$, and (44), Lemma 6 can be applied to $\tilde{s}=\tilde{p}\tilde{q}$; thus also using (38) and (40) we obtain

$$\begin{aligned}|s^{(m)}(y)|=|\tilde{s}^{(m)}(0)|&\leqq c_{24}(a,m)\left(\frac{M(l+1)^2}{\sqrt{r}}\right)^m\|\tilde{s}\|_a\\ &=c_{24}(a,m)\left(\frac{M(l+1)^2}{\sqrt{r}}\right)^m\|s\|_a \qquad (M^{-2}\leqq r\leqq M^2).\end{aligned} \tag{45}$$

By Gauss' Theorem $s^{(m)}(x)$ has all its zeros in $\{z\in\mathbb{C} \mid \operatorname{Re} z\geqq 0\}$; hence $y\leqq 0$ yields

$$|s^{(m)}(0)|\leqq|s^{(m)}(y)| \tag{46}$$

which together with (45) gives the lemma. ∎

Now let

$$\|p\|_{a,\delta}:=\sup_{\delta\leqq x<\infty}|p(x)\exp(-x^a)| \qquad (p\in\Pi_n,\ a,\delta>0). \tag{47}$$

We need the following

LEMMA 8. (a) *For all* $0 \leqq k \leqq n$, $m \geqq 1$, r, a, $\delta > 0$ *there exists a* $0 \not\equiv s^* = s^*_{n,k,m,r,\delta} \in W^k_n(r)$ *such that*

$$\frac{|s^{*(m)}(0)|}{\|s^*\|_{a,\delta}} = \sup_{s \in W^k_n(r)} \frac{|s^{(m)}(0)|}{\|s\|_{a,\delta}}. \tag{48}$$

(b) s^* *has at most* m *roots (counting multiplicities) in*

$$D_4(r) := \{z \in \mathbb{C} \setminus \mathbb{R} \mid |z - r| > r\}.$$

The proof is rather similar to that of Lemma 5 of [2], so we omit it.

Now let $\tilde{\delta} = 1/4n^2$. Then using Markov's inequality (1) on [0, 1] (with $m = 1$) and the mean value theorem, we easily obtain

$$\|p\|_a \leqq 2e \|p\|_{a,\tilde{\delta}} \qquad (p \in \Pi_n). \tag{49}$$

From now on let $s^* := s^*_{n,k,m,r,\tilde{\delta}}$. Then in the same way as in [2] (see (20)–(37) there), from Lemmas 7 and 8 and (49) we can deduce that

$$|s^{*(m)}(0)| \leqq c_{25}(a, m)((k+1)^2 L_n(a, r))^m \times \|s^*\|_{a,\tilde{\delta}} \qquad (M^{-2} \leqq r \leqq M^2) \tag{50}$$

whence because of the maximality of s^* we get

$$\begin{aligned} |s^{(m)}(0)| &\leqq c_{25}(a, m)((k+1)^2 L_n(a, r))^m \|s\|_{a,\tilde{\delta}} \\ &\leqq c_{25}(a, m)((k+1)^2 L_n(a, r))^m \|s\|_a \\ &\qquad (s \in W^k_n(r),\ M^{-2} \leqq r \leqq M^2). \end{aligned} \tag{51}$$

Now observe that $p \in W^k_n(r)$, $y \in [0, r]$ imply $s(x) := p(x+y) \in W^k_n(r/2)$; thus, applying (51) to s and using that $x^a + y^a \geqq (x+y)^a$ $(x, y \geqq 0,$ $0 < a \leqq 1)$ we obtain

$$\begin{aligned} |p^{(m)}(y)| &= |s^{(m)}(0)| \\ &\leqq c_{26}(a, m)((k+1)^2 L_n(a, r))^m \|s\|_a \\ &\leqq c_{26}(a, m)((k+1)^2 L_n(a, r))^m \exp(y^a) \\ &\quad \times \sup_{x \geqq 0} |p(x+y) \exp(-(x+y)^a)| \\ &\leqq c_{26}(a, m)((k+1)^2 L_n(a, r))^m \exp(y^a) \|p\|_a \\ &\qquad (p \in W^k_n(r),\ 0 \leqq y \leqq r,\ M^{-2} \leqq r \leqq M^2). \end{aligned} \tag{52}$$

This together wih Lemma 1 yields the theorem, when $M^{-2} \leqq r \leqq M^2$. If $0 < r \leqq M^{-2}$, then Theorem 1 gives the desired result. If $M^2 < r < \infty$, then the relation $W_n^k(r) \subset W_n^k(M^2)$ and the just proved part of the theorem yield the statement.

Case 3. $0 < a < \frac{1}{2}$. Now Theorem 1 implies Theorem 2. ∎

Proof of Theorem 3. We shall use the following infinite-finite range inequality,

$$\|f\|_a \leqq c_{27}(a) \max_{0 \leqq y \leqq c_{28}(a) n^{1/a}} |f(y) \exp(-y^a)| \quad (f \in \Pi_{2n}, 0 < a < \infty), \tag{53}$$

with suitable $c_{27}(a)$, $c_{28}(a) \geqq 1$. By using the substitutions $y = x^2$ and $b = 2a$ this is an obvious consequence of the analogous result for the interval $(-\infty, \infty)$ and weight function $\exp(-|x|^b)$ $(b > 0)$; see [7, Theorem A] or [10, Lemma 6.3]. To prove the sharpness of Theorem 2 when $k = 0$, $1 \leqq m \leqq n$, and $0 < a \neq \frac{1}{2}$, we distinguish three cases.

Case 1. $0 < r \leqq (\pi/4m)\, c_{28}(a) n^{1/a}$ if $1 \leqq a < \infty$, or $0 < r \leqq (\pi/4m) n^{2-1/a}$ if $\frac{1}{2} < a$. Let

$$x_j = \left(\frac{c_{28}(a)}{2} n^{1/a} - \frac{4m}{\pi} r\right) \cos \frac{(2n-2j+1)\pi}{2n} + \frac{c_{28}(a)}{2} n^{1/a} \qquad (1 \leqq j \leqq n), \tag{54}$$

$$z_j = x_j + ir \qquad (1 \leqq j \leqq n), \tag{55}$$

and

$$s(x) = s_{n,m,r,a}(x) = \sum_{j=1}^{n} (x - z_j)(x - \bar{z}_j) \in W_n^0(r). \tag{56}$$

By Lemma 3 of [1] and (53) we easily deduce that

$$\begin{aligned} |s(0)| &= \max_{0 \leqq x \leqq c_{28}(a) n^{1/a}} |s(x)| \\ &\geqq \max_{0 \leqq x \leqq c_{28}(a) n^{1/a}} |s(x) \exp(-x^a)| \\ &\geqq \frac{1}{c_{27}(a)} \|s\|_a. \end{aligned} \tag{57}$$

So using the notation $q(x) = \sum_{j=1}^{n} (x - x_j)$, (54)–(57), and the assumption of this case, by a simple calculation we get

$$\begin{aligned}\frac{|s^{(m)}(0)|}{\|s\|_a} &\geqq \frac{1}{c_{27}(a)} \frac{|s^{(m)}(0)|}{|s(0)|}\\ &\geqq \frac{2}{c_{27}(a)}\left(1+\frac{\pi}{4m}\right)^{-m} \frac{1}{\sqrt{2}} \frac{|q^{(m)}(0)|}{|q(0)|}\\ &\geqq \frac{\sqrt{2}}{ec_{27}(a)}\left(\sum_{j=m}^{n} \frac{1}{1-x_j}\right)^m\\ &\geqq c_{29}(a, m)(L_n(a, r))^m \qquad (1 \leqq m \leqq n).\end{aligned}$$

Case 2. $(\pi/4m)\, c_{28}(a) n^{1/a} < r < \infty$, $a \geqq 1$. Now the polynomials $s_{n,m,r,a} = x^n$ show that Theorem 2 is sharp when $k = 0$ and $1 \leqq m \leqq n$.

Case 3. $(\pi/4m)\, c_{28}(a) n^{2-1/a} \leqq r < \infty$ if $\frac{1}{2} < a < 1$ or $0 < r < \infty$ if $0 < a < \frac{1}{2}$. Now the polynomials $s_{n,m,r,a} = x$ give the desired result. ∎

Of course the sharpness of Theorem 2 when $k = 0$, $1 \leqq m \leqq n$, and $0 < a \neq \frac{1}{2}$ implies the sharpness of Theorem 1 as well.

Note 2. Theorem 2 and the examples of Theorem 3 yield that

$$c_{29}(a, r)(L_n(a, r))^m \leqq \sup \frac{\|s^{(m)}\|_a}{\|s\|_a} \leqq c_3(a.\ m)(L_n(a, r))^m$$
$$(0 \leqq r < \infty,\ 1 \leqq m \leqq n,\ 0 < a \neq \tfrac{1}{2})$$

holds not only in the case when the supremum is taken for all polynomials from $W_n^0(r)$, but for all polynomials from Π_n having all their zeros in $\{z \in \mathbb{C} \mid |\mathrm{Im}\, z| \geqq r\}$.

Proof of Theorem 4. We need

LEMMA 9. (a) *For each* $n \geqq 1$, $r \geqq 0$, a, $\delta > 0$, *and* $0 \leqq y \leqq r$ *there exists a polynomial* $p^* = p^*_{n,r,a,\delta,y} \in V_n^0(r)$ *such that*

$$\frac{|p^{*\prime}(y)|}{\|p^*\|_{a,\delta,y}} = \sup_{p \in V_n^0(r)} \frac{|p'(y)|}{\|p\|_{a,\delta,y}}, \tag{58}$$

where $\|p\|_{a,\delta,y} := \sup_{[0,\infty)\setminus(y-\delta,y+\delta)} |p(x)\exp(-x^a)|$.

(b) p^* *has all but at most one root in* $[0, r] \cup \{z \in \mathbb{C} \mid \mathrm{Re}\, z = r\}$, *and the remaining (at most one) root is in* $(-\infty, 0)$.

The proof of this lemma is rather similar to that of Lemma 5 of [2], so we omit the details.

It is easy to see that for all $a>0$, $n\geqq 1$, and $y\geqq 0$ there exists a $0<\tilde{\delta}=\tilde{\delta}(a,n,y)<1$ such that

$$\|p\|_a \leqq 2\,\|p\|_{a,\tilde{\delta},y} \qquad \text{for all} \quad p\in \Pi_n. \tag{59}$$

By Lemma 9 $p^*\in V_n^0(r)$ satisfying (58) with $\delta=\tilde{\delta}$ is of the form

$$p^*(x)=(x-x_0)^\alpha \prod_{\nu=1}^{\beta}(x-x_\nu)\left(\sum_{j=0}^{\gamma} a_j(x-r)^{2j}\right), \tag{60}$$

where

$$x_0\in(-\infty,0), \qquad \alpha=0, \qquad \text{or} \quad \alpha=1, \tag{61}$$

$$x_\nu\in[0,r] \qquad (1\leqq \nu\leqq \beta) \tag{62}$$

$$a_j\geqq 0 \qquad (0\leqq j\leqq \gamma) \tag{63}$$

and

$$\alpha+\beta+2\gamma\leqq n. \tag{64}$$

Let

$$I_1=\{j\in\mathbb{N}\,|\,0\leqq j\leqq \gamma,\ \beta+2j<2(4r+1)^a\}, \tag{65}$$

$$I_2=\{j\in\mathbb{N}\,|\,0\leqq j\leqq \gamma,\ \beta+2j\geqq (4r+1)^a\}, \tag{66}$$

and

$$p_j(x):=(x-x_0)^\alpha \prod_{\nu=1}^{\beta}(x-x_\nu)\,a_j(x-r)^{2j} \qquad (0\leqq j\leqq \gamma). \tag{67}$$

By (60), (65), (66), and (67) we have

$$p^*=f_1+f_2, \tag{68}$$

where

$$f_1:=\sum_{j\in I_1} p_j \qquad \text{and} \qquad f_2:=\sum_{j\in I_2} p_j. \tag{69}$$

By (67), (61), (62), and (66) for $j\in I_2$ and $0\leqq y\leqq r$ we obtain

$$\begin{aligned}\frac{|p_j'(y)\exp(-y^a)|}{|p_j(4r+1)\exp(-(4r+1)^a)|} &\leqq (\alpha+\beta+2j)3^{1-\beta-2j}\exp((4r+1)^a)\\ &\leqq 3(1+\beta+2j)3^{-(\beta+2j)/2}\exp((4r+1)^a-(\beta+2j)/2)\leqq c_{30}.\end{aligned} \tag{70}$$

Thus from (67), (69), (70), (68), (63), (59), and $0<\bar{\delta}<1$

$$\begin{aligned}|f_2'(y)\exp(-y^a)| &\leqq c_{30}\,|f_2(4r+1)\exp(-(4r+1)^a)| \\ &\leqq c_{30}\,|p^*(4r+1)\exp(-(4r+1)^a)| \\ &\leqq c_{30}\,\|p^*\|_{a,\bar{\delta},y} \qquad (0\leqq y\leqq r).\end{aligned} \tag{71}$$

By (69), (65), and $0\leqq \alpha\leqq 1$, f_1 is a polynomial of degree at most $l:=\min\{[2(4r+1)^a+1], n\}$, so using Theorem 1, (63), (68), and (59) we obtain

$$\begin{aligned}|f_1'(y)\exp(-y^a)| &\leqq c_2(a,1)\,K_l(a,r)\,\|f_1\|_a \\ &\leqq c_{31}(a)\,G_n(a,r)\,\|f_1\|_a \\ &\leqq c_{31}(a)\,G_n(a,r)\,\|p^*\|_a \\ &\leqq 2c_{31}(a)\,G_n(a,r)\,\|p^*\|_{a,\bar{\delta},y}\ (0<a<\infty,\, 0\leqq y<\infty).\end{aligned} \tag{72}$$

From (68), (71), and (72) we get

$$\begin{aligned}&|p^{*\prime}(y)\exp(-y^a)| \\ &\quad\leqq c_{32}(a)\,G_n(a,r)\,\|p^*\|_{a,\bar{\delta},y} \qquad (0<a<\infty,\, 0\leqq y\leqq r);\end{aligned}$$

hence the maximality of p^* yields

$$\begin{aligned}|p'(y)\exp(-y^a)| &\leqq c_{32}(a)\,G_n(a,r)\,\|p\|_{a,\bar{\delta},y} \\ &\leqq c_{32}(a)\,G_n(a,r)\,\|p\|_a \\ &\qquad (p\in V_n^0(r),\, 0<a<\infty,\, 0\leqq y\leqq r).\end{aligned} \tag{73}$$

Now let $p\in V_n^0(r)$ and $z\in[0,\infty)$ be arbitrary. Applying (73) with $y=0$ to $\tilde{p}(x):=p(x+z)\in V_n^0(r)$ and using the inequality $(x+z)^a\leqq x^a+z^a$ ($x, z\geqq 0$, $0<a\leqq 1$) we obtain

$$\begin{aligned}|p'(z)\exp(-z^a)| &= |\tilde{p}'(0)\exp(-z^a)| \\ &\leqq c_{32}(a)\,G_n(a,r)\,\|\tilde{p}\|_a\exp(-z^a) \\ &\leqq c_{32}(a)\,G_n(a,r) \\ &\qquad\times \sup_{0\leqq x<\infty}|p(x+z)\exp(-(x+z)^a)| \\ &\leqq c_{32}(a)\,G_n(a,r)\,\|p\|_a \\ &\qquad (p\in V_n^0(r),\, 0\leqq z<\infty,\, 0<a\leqq 1).\end{aligned} \tag{74}$$

If $p\in V_n^0(r)$ and $r\leqq y\leqq \frac{1}{2}n^{1/a}$, then (cf. (5)) $s:=pq_{n,y}\in \Pi_{(2[a]+1)n}$ has all its zeros outside the circle with diameter $[y, n^{1/a}]$; thus s is of the form

$$s(x)=\sum_{v=0}^{d} b_v(x-y)^v\,(n^{1/a}-x)^{d-v}$$

with $b_v\geqq 0$ $(1\leqq v\leqq d)$ and $d=(2[a]+1)n$; thus a theorem of G. G. Lorentz (see Theorem A of [6]) and (5) yield

$$\begin{aligned}|s'(y)| &\leqq \frac{c_{33}(a)n}{n^{1/a}-y}\max_{y\leqq x\leqq n^{1/a}}|s(x)|\\ &\leqq c_{34}(a)n^{1-1/a}\max_{y\leqq x\leqq n^{1/a}}|p(x)\exp(-x^a)|\\ &\leqq c_{34}(a)n^{1-1/a}\,\|p\|_a \qquad (1\leqq a<\infty,\ r\leqq y\leqq \tfrac{1}{2}n^{1/a}).\end{aligned}\tag{75}$$

Hence and from (7)

$$\begin{aligned}|p'(y)\exp(-y^a)| &\leqq |s'(y)|+|p(y)\,q'_{n,y}(y)|\\ &\leqq c_{35}(a)n^{1-1/a}\,\|p\|_a\\ &\qquad (p\in V_n^0(r),\ 1\leqq a<\infty,\ r\leqq y\leqq \tfrac{1}{2}n^{1/a}).\end{aligned}\tag{76}$$

By Lemma 1 we have

$$\begin{aligned}|p'(y)\exp(-y^a)| &\leqq c_{36}(a)\frac{n^{1-1/(2a)}}{\sqrt{y}}\,\|p\|_a\\ &\leqq c_{37}(a)n^{1-1/a}\,\|p\|_a\\ &\qquad (p\in \Pi_n,\ \tfrac{1}{2}<a<\infty,\ \tfrac{1}{2}n^{1/a}\leqq y<\infty).\end{aligned}\tag{77}$$

Finally we have

$$\|p'\|_a\leqq c_{38}(a)\,\|p\|_a \qquad (p\in\Pi_n,\ 0<a<\tfrac{1}{2})\tag{78}$$

(see Theorem 2 of [11] and Theorem 1). Now (73), (74), (76), (77), and (78) yield the theorem when $m=1$. From this, using Gauss' theorem, by induction on m we immediately obtain the desired result for all $m\geqq 1$. ∎

Proof of Theorem 5. Let $T_k(x)=\cos(k\arccos x)$ be the Chebyshev polynomial of degree k and let

$$R:=\min\{r,n^{1/a}\},\qquad a>\tfrac{1}{2},\tag{79}$$

$$p_k(x):=T_k\left(\frac{2x}{R}-1\right)\in V_n^0(r),\tag{80}$$

where

$$k:=\left[\frac{R^a}{c_{28}(a)^a}\right]\leqq n\tag{81}$$

with $c_{28}(a)\geqq 1$ defined by (53). Then using (53), (79), (80), and (81), by a simple calculation we obtain

$$\begin{aligned}\|p_k^{(m)}\|_a &\geqq |p_k^{(m)}(0)| \\ &\geqq c_{38}(m)\left(\frac{2k^2}{R}\right)^m \max_{0\leqq x\leqq R} |p_k(x)| \\ &\geqq c_{39}(a,m)(k^{2-1/a})^m \max_{0\leqq x\leqq c_{28}(a)k^{1/a}} |p_k(x)| \\ &\geqq c_{40}(a,m)(1+\min\{r^{2a-1}, n^{2-1/a}\})^m \|p_k\|_a \\ &\qquad (\tfrac{1}{2}<a<\infty,\ k\geqq m+1). \end{aligned} \tag{82}$$

Further, for the polynomials $P_n(x) := x^n \in V_n^0(0) \subset V_n^0(r)$ $(r \geqq 0)$ we have

$$\|P_n^{(m)}\|_a \geqq \begin{cases} c_{41}(a,m)(n^{1-1/a})^m \|P_n\|_a & (1\leqq a<\infty,\ n\geqq m+1) \\ c_{42}(a,m)\|P_n\|_a & (0<a<\infty,\ n=m+1). \end{cases} \tag{83}$$

Now (82) and (83) give the desired result. ∎

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